CIMPA SUMMER SCHOOL
AGAHF 2005

13-25 June 2005, Galatasaray University, Istanbul

SCIENTIFIC PROGRAMME

Speaker: Daniel Allcock `allcock@math.utexas.edu`
Title: Real hyperbolic geometry in moduli problems

We will discuss discrete groups acting on real hyperbolic space, emphasizing the isometry groups of Z-lattices of signature (n, 1). We will cover Vinberg's method of determining when such groups are generated by reflections, and how to understand the groups when they are. Then we will study some particular discrete groups acting on hyperbolic three-space, namely the ones uniformizing the components of the moduli space of real sextics in P1, and the the one uniformizing their union. This is very closely analogous to work of Carlson, Toledo and the speaker on moduli of real cubic surfaces, and displays all the interesting new phenomena found in that moduli space. The difference is that three dimensions are visualizable, so the constructions and the final answer are easier to understand than in the cubic-surface case. In order to explain the construction we will also discuss the uniformization of the moduli space of 6-tuples in the complex projective line by the complex 3-ball, which was discovered by Deligne-Mostow.

Speaker: Igor Dolgachev `idolga@umich.edu`
Title: Moduli spaces as ball quotients

2 lectures giving a very elementary introduction starting from the classical one-dimensional case and also introduce periods of K3 surfaces. I will leave Kondo to talk about moduli space of surfaces.

Speaker: Rolf Peter Holzapfel `holzapfl@mathematik.hu-berlin.de`
Title: Orbital Varieties and Invariants (4 hrs)

Complex solutions of systems of partial differential equations are multivalued in general. For nice Fuchsian systems on a manifold X the multivalence can be described by a discrete group lattice Γ, called monodromy group, acting on the uniformizing domain $\mathbb{B}$, say a hermitian symmetric one. On special subvarieties W the restricted system may have the same uniformizing quality with a subdomain $\mathbb{D}$ of $\mathbb{B}$ and discrete lattice $\Gamma_{\mathbb{D}} \subset \Gamma$ acting on

$\mathbb{D}$. There are nice cases, where $\mathbb{D}$ is a geodesic subvariety of $\mathbb{B}$. This property goes down to the embedding $W \supseteq \mathbb{D}/\Gamma \subset \mathbb{B}/\Gamma \subseteq X$ we started with.

Following also an idea of M. Uludag we define - a littlebit finer than usual - an *orbital variety* $\underline{X}$ as orbifold X (in the usual sense) together with a well-choosen weight map $w : X \to \mathbb{N}_+ \cup \{\infty\}$. Moreover we define *relative* (or *embedded*) orbital varieties $\underline{W}$. In both, the absolute and relative, cases we introduce also several types of morphisms compatible with the structures. With these notions we are able to give general definitions for *absolute* and *relative orbital invariants*.

With this language we are able to present relations between them, generalizing Hirzebruch-Mumford Proportionality to arithmetic groups with torsion. For Picard and Hilbert modular spaces the orbital invariants can be expressed by special values of Dedekind zeta functions. In complex dimension 2 (and I guess also in higher dimensions) they can be also described by explicitly known universal rational functions in the geometric data and weights, as well in the absolute as in the relative (embedded curve) cases. Elliptic modular forms of Hirzebruch-Zagier and Kudla-Cogdell type are interpreted as relative orbital invariants. This result is useful for a purely geometric determination of them after fine surface classifications.

Speaker: Michel Jambu `jambu@unice.fr`
Title: Arrangements of Hyperplanes (5 hrs)

1. Combinatorics 2. Algebras (Orlik-Solomon algebra) 3. Cohomology algebra 4. The NBC-complex 5. The nbc-set

---

Speaker: A. Kochubei:
Title: Hypergeometric functions and Carlitz differential equations over function fields

The lectures will present a survey of recent results in analysis of additive functions over function fields motivated by applications to various classes of special functions including Thakur's hypergeometric function. We will consider basic notions and results of calculus, analytic theory of differential equations with Carlitz derivatives (including a counterpart of regular singularity), umbral calculus, holonomic modules over the Weyl-Carlitz ring.

Speaker: Shigeyuki Kondo `kondo@math.nagoya-u.ac.jp`
Title: Complex ball uniformizations of the moduli spaces of del Pezzo surfaces (1-2 hrs)

I shall give complex ball uniformizations of the moduli spaces of del Pezzo surfaces via periods of K3 surfaces. Furthermore I shall mention a relation with Deligne-Mostow's complex ball uniformization of the moduli of points on $\mathbb{P}^1$, and an application of Borcherds theory of automorphic forms on type IV symmetric domain to our situation.

Speaker: Edward Looijenga looijeng@math.uu.nl
Title: Hypergeometric functions associated to arrangements (4 hrs)

Dunkl systems, Hypergeometric functions associated to Dunkl systems Projective orbifold completions (of Fubini-Study, Euclidean and ball quotient type), Deligne-Mostow's theory as a special case. Modular interpretation of some special cases

---

Speaker: Keiji Matsumoto matsu@math.sci.hokudai.ac.jp
Title: Invariant functions with respect to the Whitehead link (1-2 hrs)

It is known that the complement of the Whitehead link admits a hyperbolic structure. This space can be expressed as $\mathbb{H}^3/W$, where $\mathbb{H}^3$ is the real 3-dimensional upper half space, and the discreate subgroup W of $GL_2(\mathbb{C})$, generated by two elements $\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1+i & 1 \end{pmatrix}$, is isomorphic to the fundamental group of the complement of the Whitehead link. I shall represent the quotient space $\mathbb{H}^3/W$ by constructing invariant functions on $\mathbb{H}^3$ with respect to W in terms of theta functions on the bounded symmetric domain of type $I_{2,2}$.

Speaker: Hironori Shiga shiga@math.s.chiba-u.ac.jp
Title: Hypergeometric functions and arithmetic geometric means

On the recent joint work with Kenji Koike, containing an introduction lecture.

1. Revue of the classical theory of Gauss' AGM and some variants
Story that AGM = elliptic integral = ${}_2F_1$.
The Gauss hypergeometric function and the Appell hypergeometric function F_1
Story of the isogeny formula of theta constants.
The Jacobi theta functions and the Riemann theta functions.
Story of two variants of AGM by Borweins and Richelot.

2. The Schwarz map of $HGDF$
The Schwarz map of the Gauss $HGDF$ and the triangle monodromy group $\Delta(\nu_0, \nu_1, \nu_\infty)$
List of arithmetic triangle groups given by K. Takeuchi
Picard modular function and its representation in terms of Riemann theta constants (As a example of the modular function derived from the list of Terada and Deligne-Mostow)
List of T and D-M
An isogeny between the Jacobi varieties of Picard curves

3. A new three terms AGM
Explanation of our result on three terms AGM using all the preceeding stories. That is a generalization of the cubic AGM discovered by Borweins.

References

J.M. Borwein and P.B.Borwein, *A cubic counterpart of Jacobi's identity and the AGM,* Trans. Amer. Math. Soc., 323 (1991), no. 2, 691–701.

J.B. Bost and J.F. Mestre, *Moyenne Arithmético-Géoétrique et Périodes des Courbes de Genre 1 et 2,* Gaz. Math. No. 38 (1988), 36–64.

K. Koike and H. Shiga, *A three terms Arithmetic-Geometric mean,* Tech. Rep. Chiba univ. 2005.

K. Koike and H. Shiga, *Isogeny formulas for the Picard modular forms and a generalization of the cubic AGM ,* Tech. Rep. Chiba univ. 2005.

F. Richelot, *Essai sur une méthode générale pour déterminer la valeur des intégrales ultra-elliptiques, fondée sur des transformations remarquables de ces transcendantes .* C.R.Acad. Sci. Paris,2(1836),622-627.

H. Shiga, *On the representation of the Picard modular function byθ constants I-II,* Pub. R.I.M.S. Kyoto Univ. 24(1988), 311-360.

M. Yoshida, *Fuchsian differential equations, Aspects of Mathematics,* Vieweg, Braunschweig(1987).

Speaker: Jan Stienstra `stien@math.uu.nl`

Title: Gel'fand-Kapranov-Zelevinsky hypergeometric systems and their role in mirror symmetry and in string theory (4hrs)

The lectures are based on a selection of topics from the author's paper "Resonant hypergeometric systems and mirror symmetry" (alg-geom/9711002) streamlined, updated and extended with ideas from more recent research (including modular aspects).

Speaker: Toshiaki Terada, `terada@belle.shiga-med.ac.jp`

Title: Hypergeometric representation of the group of pure braids.

The linear representation of the fundamental group of the domain of definition of Lauricella's hypergeometric function and its faithfulness.

Speaker: A. Muhammed Uludağ muludag@gsu.edu.tr
Title: Geometry of Complex Orbifolds (3 hrs)

Definitions of orbifolds, orbifold geometry in complex dimension 1, Theorems of Fox-Bundgaard-Nielsen on orbifold uniformization in dimension 1. Orbifold geometry in dimension 2, Orbifolds over the Apollonius configuration, ball quotient orbifold towers. Some examples of K3 orbifolds. Orbifold geometry in higher dimensions. Braid groups of an orbifold.

Speaker: Alexander Varchenko, anv@email.unc.edu
Title: Special functions, KZ type equations, and representation theory (6 hrs)

Speaker: Jürgen Wolfart wolfart@math.uni-frankfurt.de
Title: Arithmetic of Schwarz maps

For the Gauss hypergeometric differential equations with certain rational parameters, Schwarz maps are just the classical triangle functions. Under which conditions do they take algebraic values at algebraic arguments? We will see that the integral representation of hypergeometric functions provides an interpretation of Schwarz maps as period quotients on families of abelian varieties, and the number theoretic question has at least partial answers given on the one hand by automorphic functions and on the other hand by modern transcendence theory. The techniques explained here generalize to other kinds of Schwarz maps as well (with apparent singularities or in several variables). The lectures will cover old and recent joint work with H. Shiga and in part with T. Tsutsui.

Speaker: Masaaki Yoshida myoshida@math.kyushu-u.ac.jp
Title: Schwarz maps (1-3 hrs)

In the 20th century, various generalizations of the Schwarz map (of the hypergeometric differential equation) are studied, under the condition that the exponents are real. I would like to propose a study of the Schwarz map when the exponents are not necessarily real. I would like to start from the binomial theorem the exponential function, the logarithmic function, the power function, its detailed (elementary) study, and comes some fundamental facts on hypergeometric function, which can be found in most books, and finaly the main topic: Schwarz map, old and new. This (at least first half or more) is intended to give a basic knowledge to the audience who are not familiar with the hypergeometric functions and automorphic functions.
